Calcul intégral — $2^{ème}$ Bac Pro

Correction des exercices du polycopié du cours

EX.1 Vérifier les calculs suivants :

1°/ $\int_{-1}^{0}(2x+1)\,dx = 0$

2°/ $\int_{0}^{1}(x^3+6x+1)\,dx = \frac{13}{4}$

3°/ $\int_{4}^{16}\frac{dx}{2\sqrt{x}} = 2$; 4°/ $\int_{0}^{3}\frac{dx}{x+2} = \ln\left(\frac{5}{2}\right)$

solution : 1°/ $\int_{-1}^{0}(2x+1)\,dx \overset{?}{=} 0$

on a : $\int_{-1}^{0}(2x+1)\,dx = \int_{-1}^{0} f(x)\,dx$

avec : $f(x) = 2x+1$

une primitive de f : $F(x) = x^2+x$

car : $F'(x) = (x^2+x)' = 2x+1$

donc : $\int_{-1}^{0}(2x+1)\,dx = \int_{-1}^{0}F'(x)\,dx$

$= [F(x)]_{-1}^{0} = [x^2+x]_{-1}^{0}$

$= 0^2+0-((-1)^2+(-1)) = -(1-1) = \boxed{0}$

2°/ $\int_{0}^{1}\underbrace{(x^2+6x+1)}_{f(x)}\,dx \overset{?}{=} \frac{13}{4}$

$\Rightarrow F(x) = \frac{x^3}{3}+6\frac{x^2}{2}+x$; F est une primitive.

donc :

$\int_{0}^{1} f(x)\,dx = [F(x)]_0^1 = F(1)-F(0)$

avec : $\begin{cases} F(1) = \frac{1}{3}+\frac{6}{2}+1 = \frac{1}{3}+4 = \frac{13}{3} \\ F(0) = 0+0+0 = 0 \end{cases}$

donc : $\int_{0}^{1}(x^2+6x+1)\,dx = \frac{13}{3}$

3°/ $\int_{4}^{16}\frac{dx}{2\sqrt{x}} = \int_{4}^{16}\underbrace{\frac{1}{2\sqrt{x}}}_{f(x)}\,dx \overset{?}{=} 2$

$f(x) = \frac{1}{2\sqrt{x}} \Rightarrow F(x) = \sqrt{x}$

donc : $\int_{4}^{16} f(x)\,dx = [\sqrt{x}]_4^{16} = \sqrt{16}-\sqrt{4}$

$= 4-2 = \boxed{2}$

4°/ $\int_{0}^{3}\frac{dx}{x+2} = \int_{0}^{3}\frac{1}{x+2}\,dx \overset{?}{=} \ln\left(\frac{5}{2}\right)$

$f(x) = \frac{1}{x+2} = \frac{(x+2)'}{x+2} = \ln'(x+2)$

$F(x) = \ln(x+2)$ donc :

$\int_{0}^{3}\frac{dx}{x+2} = F(x)]_0^3 = [\ln(x+2)]_0^3$

$= \ln(3+2)-\ln(0+2) = \ln 5 - \ln 2$

$= \boxed{\ln\left(\frac{5}{2}\right)}$

EX.2 Montrer que :

1°/ $\int_{0}^{\pi}\left(4x+\frac{2}{3}\sin(x)\right)dx = 2\pi^2+\frac{4}{3}$

2°/ $\int_{0}^{1}(5x^3+e^x)\,dx = e+\frac{1}{4}$

3°/ $\int_{1}^{e}\left(\frac{1}{x}-\frac{1}{x^2}\right)dx = \frac{1}{e}$

4°/ $\int_{1}^{2}(2x+3)(x^2+3x)^2\,dx = 312$

Solution : 1°/ $\int_{0}^{\pi}\left(4x+\frac{2}{3}\sin(x)\right)dx$

$= 4\times\int_{0}^{\pi}x\,dx+\frac{2}{3}\int_{0}^{\pi}\sin(x)\,dx$

$= 4\left[\frac{x^2}{2}\right]_0^{\pi}+\frac{2}{3}[-\cos(x)]_0^{\pi}$

$= 4\left[\frac{\pi^2}{2}-0\right]+\frac{2}{3}[-(-1)+1]$

$= 4\times\frac{\pi^2}{2}+\frac{2}{3}\times 2 = \boxed{2\pi^2+\frac{4}{3}}$

2°/ $\int_{0}^{1}(5x^3+e^x)\,dx = 5\times\int_{0}^{1}x^3\,dx+\int_{0}^{1}e^x\,dx$

$= 5\left[\frac{x^4}{4}\right]_0^1+[e^x]_0^1 = 5\times\left(\frac{1}{4}-0\right)+e^1-e^0$

$$\int_0^1 (5x^3+e^x)\,dx = \frac{5}{4}+e-1$$
$$= e+\frac{5}{4}-1 = e+\frac{1}{4}$$

3°/ $\int_1^e \frac{1}{x}-\frac{1}{x^2}\,dx = \int_1^e \frac{1}{x}\,dx + \int_1^e -\frac{1}{x^2}\,dx$

$$= \left[\ln(x)\right]_1^e + \left[\frac{1}{x}\right]_1^e$$
$$= \ln(e)-\ln(1)+\frac{1}{e}-\frac{1}{1}$$
$$= 1-0+\frac{1}{e}-1 = \boxed{\frac{1}{e}}$$

car: $\ln'(x)=\frac{1}{x}$ et $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

donc: $\frac{1}{x}-\frac{1}{x^2} = \left(\ln(x)+\frac{1}{x}\right)'$

4°/ $\int_1^2 \underbrace{(2x+3)\,(x^2+3x)^2}_{f(x)}\,dx$

$$f(x) = (x^2+3x)'(x^2+3x)^2$$
$$= \frac{1}{3}\times\left[3\,(x^2+3x)'(x^2+3x)^2\right]$$
$$= \frac{1}{3}F'(x)\ ; \text{ avec: } F(x) = (x^2+3x)^3$$

donc: $\int_1^2 f(x)\,dx = \frac{1}{3}\left[F(x)\right]_1^2 = \frac{1}{3}\left(F(2)-F(1)\right)$

$$\begin{cases} F(2) = (2^2+3\times 2)^3 = (4+6)^3 = 1000 \\ F(1) = (1+3)^3 = 4^3 = 64 \end{cases}$$

$$\int_1^2 f(x)\,dx = \frac{1}{3}\left[1000-64\right] = \frac{936}{3} = \boxed{312}$$

EX.3 Montrer que:

1°/ $\int_0^3 2x\,e^{x^2}\,dx = e^9-1$

2°/ $\int_1^2 \sqrt{x}\,dx = \frac{4\sqrt{2}}{3}-\frac{2}{3}$

3°/ $\int_1^2 \frac{2x+3}{x^2+3x}\,dx = \ln\left(\frac{5}{2}\right)$

Solution: 1°/ $\int_0^3 2x\,e^{x^2}\,dx$

$$f(x) = (x^2)'e^{x^2} = \left(e^{x^2}\right)' = F'(x)$$

avec: $F(x) = e^{x^2}$ donc:

$$\int_0^3 f(x)\,dx = \left[F(x)\right]_0^3 = F(3)-F(0)$$
$$= e^{3^2}-e^{0^2} = e^9-e^0 = \boxed{e^9-1}$$

2°/ $\int_1^2 \sqrt{x}\,dx = \int_1^2 x^{1/2}\,dx$

$$= \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^2 = \left[\frac{x^{\frac{1}{2}+1}}{3/2}\right]_1^2$$
$$= \frac{2^{\frac{1}{2}+1}}{3/2}-\frac{1}{3/2} = \frac{2^{1/2}\times 2^1}{3/2}-\frac{2}{3}$$
$$= \sqrt{2}\times 2\times\frac{2}{3}-\frac{2}{3} = \boxed{\frac{4}{3}\sqrt{2}-\frac{2}{3}}$$

3°/ $\int_1^2 \frac{2x+3}{x^2+3x}\,dx = \int_1^2 \frac{(x^2+3x)'}{x^2+3x}\,dx$

$$= \int_1^2 \ln'(x^2+3x)\,dx = \left[\ln(x^2+3x)\right]_1^2$$
$$= \ln(2^2+3\times 2)-\ln(1+3\times 1)$$
$$= \ln(10)-\ln(4) = \ln\left(\frac{10}{4}\right) = \boxed{\ln\left(\frac{5}{2}\right)}$$

EX.4 A l'aide d'une intégrale par partie montrer que:

1°/ $\int_{-1}^2 x\,e^x\,dx = 0$

2°/ $\int_0^{\pi} x\sin(x)\,dx = \pi$

3°/ $\int_2^e 4x^3\ln(x)\,dx = 3\frac{e^4}{4}-16\ln(2)+4$

Solution:

1°/ $\int_{-1}^2 x\,e^x\,dx = \int_{-1}^2 u(x)\,v'(x)\,dx$

avec: $\begin{cases} u(x)=x \\ v'(x)=e^x \end{cases}$ donc: $\begin{cases} u'(x)=1 \\ v(x)=e^x \end{cases}$

par suite : $\int u(x)v'(x)\,dx = [u(x)v(x)]$

$- \int u'(x)v(x)\,dx$

$\Rightarrow \int_{-1}^{2} xe^x\,dx = [1 \times e^x]_{-1}^{2} - \int_{-1}^{2} 1 \times e^x\,dx$

$= e^2 - e^{-1} - \int_{-1}^{2} e^x\,dx$

$= e^2 - e^{-1} - [e^x]_{-1}^{2} = e^2 - e^{-1} - (e^2 - e^{-1})$

$= e^2 - e^{-1} - e^2 + e^{-1} = \boxed{0}$

2°/ $\int_0^{\pi} x\sin(x)\,dx = \int_0^{\pi} \underbrace{x}_{u(x)} \times \underbrace{\sin(x)}_{v'(x)}\,dx$

$\begin{cases} u(x) = x \\ v'(x) = \sin(x) \end{cases} \Rightarrow \begin{cases} u'(x) = 1 \\ v(x) = -\cos(x) \end{cases}$

donc

$\int u(x)v'(x)\,dx = [u(x)v(x)] - \int u'(x)v(x)\,dx$

$\int_0^{\pi} x\sin(x)\,dx = [-x\cos(x)]_0^{\pi} - \int_0^{\pi} 1 \times (-\cos(x))\,dx$

$= [-\pi(-1) + 0] + \int_0^{\pi} \cos(x)\,dx$

$= \pi + [\sin(x)]_0^{\pi} = \pi + (0 - 0) = \boxed{\pi}$

3°/ $\int_2^{e} 4x^3\ln(x)\,dx = \int_2^{e} (4x^3)\ln(x)\,dx$

$\begin{cases} u'(x) = 4x^3 \\ v(x) = \ln(x) \end{cases} \Rightarrow \begin{cases} u(x) = x^4 \\ v'(x) = \frac{1}{x} \end{cases}$

$\int_2^{e} 4x^3\ln(x)\,dx = [u(x)v(x)]_2^{e} - \int_2^{e} u(x)v'(x)\,dx$

$= [x^4\ln(x)]_2^{e} - \int_2^{e} x^4 \times \frac{1}{x}\,dx$

$= e^4\ln(e) - 2^4\ln(2) - \int_2^{e} x^3\,dx$

$= e^4 \times 1 - 16\ln(2) - \left[\frac{x^4}{4}\right]_2^{e}$

$= e^4 - 16\ln(2) - \left(\frac{e^4}{4} - \frac{2^4}{4}\right)$

$= e^4 - \frac{e^4}{4} - 16\ln(2) + \frac{16}{4}$

$= \left(1 - \frac{1}{4}\right)e^4 - 16\ln(2) + 4$

$= \boxed{\frac{3}{4}e^4 - 16\ln(2) + 4}$